

A (HUMAN) PROOF OF A TRIPLE BINOMIAL SUM SUPERCONGRUENCE

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ABSTRACT. In a recent article, Apagodu and Zeilberger discuss some applications of an algorithm for finding and proving congruence identities (modulo primes) of indefinite sums of many combinatorial sequence. At the end they propose some supercongruences as conjectures. Here we prove one of them and we leave some remarks for the others.

1. INTRODUCTION

We will show that

Theorem 1. *Let $p > 2$ be a prime, and let r, s, t be any positive integers, then*

$$\sum_{m_1=0}^{rp-1} \sum_{m_2=0}^{sp-1} \sum_{m_3=0}^{tp-1} \binom{m_1 + m_2 + m_3}{m_1, m_2, m_3} \equiv_{p^3} \sum_{m_1=0}^{r-1} \sum_{m_2=0}^{s-1} \sum_{m_3=0}^{t-1} \binom{m_1 + m_2 + m_3}{m_1, m_2, m_3}. \quad (1)$$

The above supercongruence appears as Conjecture 6' in [1]. After a preliminary section in which we collect some useful results, we prove such conjecture in the third section. In the final section we provide some remarks.

2. PRELIMINARY RESULTS

For a $r > 0$, let $\mathbf{s} = (s_1, \dots, s_r) \in (\mathbb{Z}^*)^r$ and let $x \in \mathbb{R}$. We define the multiple sum

$$H_n(\mathbf{s}; x) = \sum_{1 \leq k_1 < \dots < k_r \leq n} \prod_{i=1}^r \frac{x_i^{k_i}}{k_i^{|s_i|}} \quad \text{with} \quad x_i = \begin{cases} x & \text{if } s_i < 0, \\ 1 & \text{if } s_i > 0. \end{cases}$$

The number $l(\mathbf{s}) := r$ is called the depth (or length) and $|\mathbf{s}| := \sum_{j=1}^r |s_j|$ is the weight of the multiple sum. By convention, these sums are zero if $n < r$. $H_n(\mathbf{s}; 1)$ is the *ordinary multiple harmonic sum* and in that case we will simply write $H_n(\mathbf{s})$. Then it is known

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(see [5, Sections 1 and 7]) that for $p > 3$,

$$H_{p-1}(1) \equiv_{p^2} 0, \quad (2)$$

$$H_{p-1}(2) \equiv_p 0, \quad (3)$$

$$H_{p-1}(1, 1) \equiv_p 0, \quad (4)$$

$$H_{p-1}(-1; 2) \equiv_{p^2} -2q_p(2), \quad (5)$$

$$H_{p-1}(-1; 1/2) \equiv_p q_p(2), \quad (6)$$

$$H_{p-1}(-2; -1) \equiv_p 0, \quad (7)$$

$$H_{p-1}(-2; 2) \equiv_p -q_p^2(2), \quad (8)$$

$$H_{p-1}(1, -1; -1) \equiv_p q_p^2(2), \quad (9)$$

$$H_{p-1}(1, -1; 2) \equiv_p 0, \quad (10)$$

$$H_{p-1}(-1, 1; 1/2) \equiv_p 0, \quad (11)$$

where $q_p(2) = (2^{p-1} - 1)/p$. In the next section we will need the following results.

Lemma 1. *Let $p > 2$ be a prime, then we have*

$$\sum_{k=1}^{p-1} \frac{1}{k2^k} \sum_{j=1}^{k-1} \frac{2^j}{j} \equiv_p 0, \quad (12)$$

and

$$\sum_{k=1}^{p-1} \frac{2^k}{k} \sum_{j=1}^{k-1} \frac{1}{j2^j} \equiv_p -2q_p^2(2). \quad (13)$$

Proof. By (11),

$$\begin{aligned} \sum_{k=1}^{p-1} \frac{1}{k2^k} \sum_{j=1}^{k-1} \frac{2^j}{j} &\equiv_p \sum_{k=1}^{p-1} \frac{2^{-k}}{k} \sum_{j=1}^{k-1} \frac{2^{k-j}}{k-j} = \sum_{k=1}^{p-1} \sum_{j=1}^{k-1} \frac{2^{-j}}{k(k-j)} \\ &= \sum_{j=1}^{p-2} \frac{2^{-j}}{j} \sum_{k=j+1}^{p-1} \left(\frac{1}{k-j} - \frac{1}{k} \right) = \sum_{j=1}^{p-2} \frac{2^{-j}}{j} \sum_{k=1}^{p-j-1} \frac{1}{k} - H_{p-1}(-1, 1; 1/2) \\ &= \sum_{j=1}^{p-2} \frac{2^{-j}}{j} \sum_{k=j+1}^{p-1} \frac{1}{p-k} - H_{p-1}(-1, 1; 1/2) \equiv_p -2H_{p-1}(-1, 1; 1/2) \equiv_p 0. \end{aligned}$$

As regards (13), it follows from

$$H_{p-1}(-1; 2) \cdot H_{p-1}(-1; 1/2) = H_{p-1}(2) + \sum_{k=1}^{p-1} \frac{1}{k2^k} \sum_{j=1}^{k-1} \frac{2^j}{j} + \sum_{k=1}^{p-1} \frac{2^k}{k} \sum_{j=1}^{k-1} \frac{1}{j2^j},$$

after using (3), (5), (6), and (12). □

Lemma 2. *Let $p > 2$ be a prime, then we have*

$$\sum_{k=2}^{p-1} \frac{1}{k^2} \sum_{m=k}^{p-1} \frac{(-1)^{m-k}}{\binom{m}{k}} \equiv_p -2q_p(2). \quad (14)$$

Proof. We have that

$$\begin{aligned} \sum_{k=2}^{p-1} \frac{1}{k^2} \sum_{m=k}^{p-1} \frac{(-1)^{m-k}}{\binom{m}{k}} &= \sum_{k=2}^{p-1} \frac{1}{k^2} \sum_{m=0}^{p-1-k} \frac{(-1)^m}{\binom{k+m}{m}} = \sum_{k=1}^{p-2} \frac{1}{(p-k)^2} \sum_{m=0}^{k-1} \frac{(-1)^m}{\binom{p-k+m}{m}} \\ &\equiv_p \sum_{k=1}^{p-2} \frac{1}{k^2} \sum_{m=0}^{k-1} \frac{1}{\binom{k-1}{m}} = \sum_{k=1}^{p-2} \frac{1}{k2^k} \sum_{j=1}^k \frac{2^j}{j} \\ &= \sum_{k=1}^{p-1} \frac{1}{k2^k} \sum_{j=1}^{k-1} \frac{2^j}{j} + H_{p-1}(2) - \frac{H_{p-1}(-1; 2)}{(p-1)2^{p-1}} \\ &\equiv_p 0 + 0 - 2q_p(2) = -2q_p(2) \end{aligned}$$

where we used (5), (12),

$$(-1)^m \binom{p-k+m}{m} = \frac{1}{m!} \prod_{j=k-m}^{k-1} (p-j) \equiv_p \binom{k-1}{m},$$

and the identity [2, (2.4)]

$$\sum_{m=0}^{k-1} \frac{1}{\binom{k-1}{m}} = \frac{k}{2^k} \sum_{j=1}^k \frac{2^j}{j}.$$

□

Lemma 3. *Let i, j be non-negative integers.*

Let $p > 2$ be a prime, then, for $0 < r < p$, we have

$$\binom{(i+j)p}{r+ip} \equiv_{p^3} \binom{i+j}{i} \binom{p}{r} j \left(1 - p \left((i+j-1)H_{r-1}(1) + \frac{i}{r} \right) \right), \quad (15)$$

and

$$\sum_{m=0}^{p-1} \binom{p-1+(i+j)p}{m+ip} \equiv_{p^3} \binom{i+j}{i} \left(1 + (i+j+1)pq_p(2) + \binom{i+j+1}{2} p^2 q_p^2(2) \right). \quad (16)$$

Proof. We have that

$$\begin{aligned}
\binom{(i+j)p}{r+ip} &= \binom{(i+j)p}{ip} \frac{jp}{ip+r} \prod_{k=1}^{r-1} \frac{jp-k}{ip+k} \\
&\equiv_{p^3} \binom{i+j}{i} \frac{jp(-1)^{j-1}}{ip+r} \prod_{k=1}^{r-1} \frac{1-\frac{ip}{k}}{1+\frac{ip}{k}} \\
&\equiv_{p^3} \binom{i+j}{i} \frac{jp(-1)^{j-1}}{r} \left(1 - \frac{ip}{r}\right) (1 - jpH_{r-1}(1)) (1 - ipH_{r-1}(1)) \\
&\equiv_{p^3} \binom{i+j}{i} \frac{jp(-1)^{r-1}}{r} \left(1 - p \left((i+j)H_{r-1}(1) + \frac{i}{r}\right)\right).
\end{aligned}$$

Congruence (15) follows as soon as we note that

$$\binom{p}{r} \equiv_{p^3} \frac{p(-1)^{r-1}}{r} (1 - pH_{r-1}(1)).$$

As regards (16),

$$\begin{aligned}
\sum_{m=0}^{p-1} \binom{p-1+(i+j)p}{m+ip} &= \sum_{m=0}^{p-1} \sum_{l=0}^{p-1} \binom{(i+j)p}{m-l+ip} \binom{p-1}{l} \\
&= \binom{(i+j)p}{ip} 2^{p-1} + \sum_{l=1}^{p-1} \binom{p-1}{l} \sum_{r=1}^l \left(\binom{(i+j)p}{r+ip} + \binom{(i+j)p}{r+jp} \right) \\
&\equiv_{p^3} \binom{i+j}{i} 2^{p-1} + \binom{i+j}{i} \sum_{l=1}^{p-1} \binom{p-1}{l} \sum_{r=1}^l \binom{p}{r} \\
&\quad \cdot \left((i+j) - p(i+j)(i+j-1)H_{r-1}(1) - \frac{2pij}{r} \right)
\end{aligned}$$

where in the last step we applied (15). By (4) and (9),

$$\begin{aligned}
\sum_{l=1}^{p-1} \binom{p-1}{l} \sum_{r=1}^l \binom{p}{r} H_{r-1}(1) &\equiv_{p^2} -p \sum_{r=1}^{p-1} \frac{(-1)^r H_{r-1}(1)}{r} \sum_{r=l}^{p-1} (-1)^l \\
&= -\frac{p}{2} (H_{p-1}(1, -1; -1) + H_{p-1}(1, 1)) \equiv_{p^2} -\frac{pq_p^2(2)}{2},
\end{aligned}$$

and similarly, by (3) and (7),

$$\sum_{l=1}^{p-1} \binom{p-1}{l} \sum_{r=1}^l \binom{p}{r} \frac{1}{r} \equiv_{p^2} -\frac{p}{2} (H_{p-1}(-2; -1) + H_{p-1}(2)) \equiv_{p^2} 0.$$

Moreover we have the identity

$$\sum_{l=1}^{p-1} \binom{p-1}{l} \sum_{r=1}^l \binom{p}{r} = 2^{p-1}(2^{p-1} - 1).$$

Finally, we obtain

$$\sum_{m=0}^{p-1} \binom{p-1+(i+j)p}{m+ip} \equiv_{p^3} \binom{i+j}{i} \left(2^{p-1} + (i+j)2^{p-1}(2^{p-1} - 1) + p^2 \binom{i+j}{2} q_p^2(2) \right)$$

which is equivalent to (16). \square

Lemma 4. *Let i, j be non-negative integers.*

Let $p > 2$ be a prime, then for $0 \leq m \leq k < p$, we have

$$\binom{k+(i+j)p}{m+ip} \equiv_{p^2} \binom{i+j}{i} \binom{k}{m} (1 + p((i+j)H_k(1) - jH_{k-m}(1) - iH_m(1))), \quad (17)$$

and

$$\sum_{m=0}^k \binom{k+(i+j)p}{m+ip} \equiv_{p^2} 2^k \binom{i+j}{i} \left(1 + p(i+j) \sum_{m=1}^k \frac{1}{m2^m} \right). \quad (18)$$

Proof. We have that,

$$\begin{aligned} \binom{k+(i+j)p}{m+ip} &= \sum_{l=0}^k \binom{(i+j)p}{m-l+ip} \binom{k}{l} \\ &= \binom{i+j}{j} \binom{k}{m} + \sum_{l=1}^m \binom{(i+j)p}{l+ip} \binom{k}{m-l} + \sum_{l=1}^{k-m} \binom{(i+j)p}{l+jp} \binom{k}{m+l} \end{aligned}$$

Then we apply (15) modulo p^2 ,

$$\begin{aligned} \binom{k+(i+j)p}{m+ip} &\equiv_{p^2} \binom{i+j}{j} \left[\binom{k}{m} + j \sum_{l=1}^m \binom{p}{l} \binom{k}{m-l} + i \sum_{l=1}^{k-m} \binom{p}{l} \binom{k}{k-m-l} \right] \\ &= \binom{i+j}{j} \left[(1-j-i) \binom{k}{m} + j \binom{p+k}{m} + i \binom{p+k}{k-m} \right] \\ &\equiv_{p^2} \binom{i+j}{i} \binom{k}{m} (1 + p((i+j)H_k(1) - jH_{k-m}(1) - iH_m(1))). \end{aligned}$$

From (17), by summing over m we obtain

$$\sum_{m=0}^k \binom{k+(i+j)p}{m+ip} \equiv_{p^2} \binom{i+j}{i} \left(2^k + p(i+j) \left(2^k H_k(1) - \sum_{m=0}^k \binom{k}{m} H_m(1) \right) \right).$$

Congruence (18) follows after applying the identity [3, (39)]

$$\sum_{m=0}^k \binom{k}{m} H_m(1) = 2^k \left(H_k(1) - \sum_{m=1}^k \frac{1}{m2^m} \right).$$

□

3. PROOF OF THE SUPERCONGRUENCE (1).

Let LHS and RHS be the left-hand side and the right-hand side of (1). We have that

$$\begin{aligned}
\text{LHS} &= \sum_{m_1=0}^{rp-1} \sum_{m_2=0}^{sp-1} \binom{m_1+m_2}{m_1} \sum_{m_3=0}^{tp-1} \binom{m_1+m_2+m_3}{m_3} \\
&= tp \sum_{m_1=0}^{rp-1} \sum_{m_2=0}^{sp-1} \binom{m_1+m_2}{m_1} \binom{m_1+m_2+tp}{m_1+m_2} \frac{1}{m_1+m_2+1} \\
&= tp \sum_{m=0}^{rp-1} \sum_{k=m}^{m+sp-1} \binom{k}{m} \binom{k+tp}{k} \frac{1}{k+1} \\
&= tp \sum_{i=0}^{r-1} \sum_{j=0}^{s-1} \sum_{m=0}^{p-1} \sum_{k=m}^{m+p-1} \binom{k+(i+j)p}{m+ip} \binom{k+(i+j+t)p}{k+(i+j)p} \frac{1}{k+(i+j)p+1} \\
&= \sum_{i=0}^{r-1} \sum_{j=0}^{s-1} (A_{ij} + B_{ij} + C_{ij})
\end{aligned}$$

where

$$\begin{aligned}
A_{ij} &:= tp \sum_{m=0}^{p-1} \sum_{k=m}^{p-1} \binom{k+(i+j)p}{m+ip} \binom{k+(i+j+t)p}{k+(i+j)p} \frac{1}{k+(i+j)p+1}, \\
B_{ij} &:= \frac{tp}{(i+j+1)p+1} \binom{(i+j+t+1)p}{(i+j+1)p} \sum_{m=1}^{p-1} \binom{(i+j+1)p}{m+ip}, \\
C_{ij} &:= tp \sum_{m=0}^{p-1} \sum_{k=1}^{m-1} \binom{k+(i+j+1)p}{m+ip} \binom{k+(i+j+t+1)p}{k+(i+j+1)p} \frac{1}{k+1+(i+j+1)p}.
\end{aligned}$$

In a similar way

$$\text{RHS} = t \sum_{i=0}^{r-1} \sum_{j=0}^{s-1} \binom{i+j}{i} \binom{i+j+t}{i+j} \frac{1}{i+j+1}.$$

By Wolstenholme's theorem $\binom{ap}{bp} \equiv_{p^3} \binom{a}{b}$, and $\binom{n_1 p + n_0}{k_1 p + k_0} \equiv_p \binom{n_1}{k_1} \binom{n_0}{k_0}$,

$$\begin{aligned} B_{ij} &\equiv_{p^3} \frac{t(i+j+1)p^2}{(i+j+1)p+1} \binom{i+j+t+1}{i+j+1} \sum_{m=1}^{p-1} \binom{p-1+(i+j)p}{m-1+ip} \frac{1}{m+ip} \\ &\equiv_{p^3} tp^2(i+j+1) \binom{i+j+t+1}{i+j+1} \binom{i+j}{i} \sum_{m=1}^{p-1} \binom{p-1}{m-1} \frac{1}{m} \\ &\equiv_{p^3} 2tp^2(i+j+t+1) \binom{i+j+t}{i, j, t} q_p(2), \end{aligned}$$

where

$$\sum_{m=1}^{p-1} \binom{p-1}{m-1} \frac{1}{m} = \frac{1}{p} \sum_{m=1}^{p-1} \binom{p}{m} = \frac{2^p - 2}{p} = 2q_p(2).$$

Finally, we note that for $k < m < p$

$$\binom{k+p}{m} = \frac{1}{m!} \prod_{j=1}^k (j+p) \cdot p \cdot \prod_{j=1}^{m-(k+1)} (p-j) \equiv_{p^2} \frac{p(-1)^{m-(k+1)}}{(k+1) \binom{m}{k+1}},$$

and

$$\begin{aligned} \binom{k+(i+j+1)p}{m+ip} &= \frac{(k+p+(i+j)p) \cdots (p+(i+j)p)}{(m+ip) \cdots (1+ip)} \binom{k+p-m+(i+j)p}{ip} \\ &\equiv_{p^2} (i+j+1) \binom{k+p}{m} \binom{k-m+p+(i+j)p}{ip} \\ &\equiv_{p^2} (i+j+1) \frac{p(-1)^{m-(k+1)}}{(k+1) \binom{m}{k+1}} \binom{i+j}{i}. \end{aligned}$$

Moreover

$$\binom{k+(i+j+t+1)p}{k+(i+j+1)p} \equiv_p \binom{i+j+t+1}{i+j+1},$$

and we obtain

$$\begin{aligned} C_{ij} &= tp^2(i+j+1) \binom{i+j+t+1}{i+j+1} \binom{i+j}{i} \sum_{m=0}^{p-1} \sum_{k=1}^{m-1} \frac{(-1)^{m-(k+1)}}{(k+1)^2 \binom{m}{k+1}} \\ &\equiv_{p^3} tp^2(i+j+t+1) \binom{i+j+t}{i, j, t} \sum_{k=1}^{p-2} \frac{1}{(k+1)^2} \sum_{m=k+1}^{p-1} \frac{(-1)^{m-(k+1)}}{\binom{m}{k+1}} \\ &\equiv_{p^3} tp^2(i+j+t+1) \binom{i+j+t}{i, j, t} \sum_{k=2}^{p-1} \frac{1}{k^2} \sum_{m=k}^{p-1} \frac{(-1)^{m-k}}{\binom{m}{k}} \\ &\equiv_{p^3} -2tp^2(i+j+t+1) \binom{i+j+t}{i, j, t} q_p(2) \end{aligned}$$

where in the last step we used (14). Therefore $B_{ij} + C_{ij} \equiv_{p^3} 0$ and it suffices to show that

$$A_{ij} \equiv_{p^3} \frac{t}{i+j+1} \binom{i+j}{i} \binom{i+j+t}{i+j}. \quad (19)$$

Now, by (16),

$$\begin{aligned} p \sum_{k=p-1}^{p-1} \sum_{m=0}^k \cdots &= \frac{1}{i+j+t+1} \binom{(i+j+t+1)p}{(i+j+1)p} \sum_{m=0}^k \binom{p-1+(i+j)p}{m+ip} \\ &\equiv_{p^3} \frac{1}{i+j+1} \binom{i+j+t}{i+j} \binom{i+j}{i} 2^{(p-1)(i+j+1)} \\ &\equiv_{p^3} \binom{i+j+t}{i, j, t} \left(\frac{1}{i+j+1} + pq_p(2) + \frac{(i+j)}{2} p^2 q_p^2(2) \right). \end{aligned}$$

because

$$\frac{(2^{p-1})^{(i+j+1)}}{i+j+1} = \frac{(1 + pq_p(2))^{i+j+1}}{i+j+1} \equiv_{p^3} \frac{1}{i+j+1} + pq_p(2) + \frac{(i+j)}{2} p^2 q_p^2(2).$$

Moreover by (17) and (18),

$$\begin{aligned} p \sum_{k=0}^{p-2} \sum_{m=0}^k \cdots &\equiv_{p^3} p \sum_{k=0}^{p-2} \left(\frac{1}{k+1} - \frac{p(i+j)}{(k+1)^2} \right) \binom{i+j+t}{i+j} (1 + p(i+j)H_k(1)) \\ &\quad \cdot 2^k \binom{i+j}{i} \left(1 + p(i+j) \sum_{m=1}^k \frac{1}{m2^m} \right) \\ &\equiv_{p^3} \frac{p}{2} \binom{i+j+t}{i, j, t} \left(H_{p-1}(-1; 2) - p(i+j)H_{p-1}(-2; 2) \right. \\ &\quad \left. + p(i+j)H_{p-1}(1, -1; 2) + p(i+j) \sum_{k=0}^{p-2} \frac{1}{k+1} \sum_{m=1}^k \frac{1}{m2^m} \right) \\ &\equiv_{p^3} \binom{i+j+t}{i, j, t} \left(-pq_p(2) - \frac{(i+j)}{2} p^2 q_p^2(2) \right), \end{aligned}$$

where in the last step we used (5), (8), (10), and (13). Hence

$$A_{ij} = tp \sum_{k=0}^{p-1} \sum_{m=0}^k \cdots \equiv_{p^3} \frac{t}{i+j+1} \binom{i+j+t}{i, j, t}.$$

and the proof of (19) is complete.

4. SOME FURTHER REMARKS

In [1] appeared some other supercongruences.

- Supercongruence 1: for any prime p ,

$$\sum_{n=0}^{p-1} \binom{2n}{n} \equiv_{p^2} \left(\frac{p}{3}\right)$$

which is congruence (1.9) at p. 647 in [4] (here $\left(\frac{p}{3}\right)$ is the Legendre symbol).

- Supercongruence 2: for any prime p ,

$$\sum_{n=0}^{p-1} C_n \equiv_{p^2} \frac{1}{2} \left(3 \left(\frac{p}{3}\right) - 1 \right)$$

which is congruence (1.7) at p. 647 in [4] (here C_n is the n th Catalan numbers).

- Supercongruence 5: for any prime p ,

$$\sum_{n=0}^{p-1} \sum_{m=0}^{p-1} \binom{n+m}{m}^2 \equiv_{p^2} \left(\frac{p}{3}\right)$$

which is equivalent to Supercongruence 1 because

$$\sum_{n=0}^{p-1} \sum_{m=0}^{p-1} \binom{n+m}{m}^2 = \sum_{k=0}^{p-1} \sum_{m=0}^k \binom{k}{m}^2 + \sum_{k=0}^{p-2} \sum_{m=k+1}^{p-1} \binom{k+p}{m}^2 \equiv_{p^2} \sum_{k=0}^{p-1} \binom{2k}{k}$$

where we used the fact that p divides $\binom{k+p}{m}$ and by Vandermonde's convolution $\sum_{m=0}^k \binom{k}{m}^2 = \binom{2k}{k}$.

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